International Journal of Heat and Mass Transfer 52 (2009) 3396–3401

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

Technical Note

# Analytical solution of Graetz problem in pipe flow with periodic inlet temperature variation

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A R T I C L E I N F O

*Article history:*
Received 15 September 2008
Accepted by 24 February 2009
Available online 3 April 2009

*Keywords:*
Graetz problem
Periodic heat transfer
Poiseuille flow
Complex eigenvalues
Kummer functions

A B S T R A C T

The paper reports an analytical solution for the temperature field in a fully developed pipe flow subject to periodic (of any shape) inlet temperature variation. The solution is given in term of a series of Kummer functions for the cases of uniform and constant wall temperature and wall heat flux, thus comprising also the adiabatic wall case. A "fully developed" region for the fluctuating component of the fluid temperature is also evidenced and closed-form solutions are given. An interpretation of the temperature field as superposition of travelling thermal waves is presented and discussed.



## 1. Introduction

The need of modelling heat exchangers and regenerators working under time varying conditions prompted the first analysis of the unsteady heat transfer in pipe flows. Sparrow and De Farias [1] were probably the first to study the effect of time varying inlet temperature conditions in a flat plate channel also for a turbulent slug flow, and pointed out the difficulties in numerically estimating the complex eigenvalues of the related Sturm_Liouville problem. Kakac and Yener [2] and Kakac [3] studied the problem posed by the analysis of transient temperature variation in parallel plate channels reporting analytical solutions. Cotta and Ozisik [4] applied a variation of the generalised integral transform technique to by-pass the analysis of complex eigenvalue problems and to solve the transient laminar forced convection problem inside parallel plate channels and tubes considering fully developed velocity distributions and sinusoidal variations of inlet temperature while Kim et al. [5] extended the solution to the case of arbitrarily shaped temporal inlet temperature variations. The method required numerical solutions of systems of algebraic or ordinary differential equations. Unsal [6,7] fully recognised the importance of solving the complex eigenvalue problem and gave an approximate analytical solution for the case of laminar flow between flat plates and in cylindrical pipes applying the method of matched asymptotic expansion: two asymptotic expansion of the eigenfunctions, one valid near the channel centreline and the other valid near the wall, are combined yielding a single uniformly valid composite asymptotic expansion for the complex eigenfunctions. He also presented another way to solve the complex eigenvalue problem which is accurate for the smaller eigenvalues.

Following the work of Unsal [7] the present paper reports an analytical solution of the problem of periodic (arbitrarily shaped) time varying inlet temperature in fully developed pipe-flow based on a generalised Fourier expansion in term of Kummer functions, thus overcoming the approximation inherent to the asymptotic expansion. The solution is applicable to the cases of uniform and constant wall temperature and wall heat flux, thus comprising also the adiabatic wall case.

## 2. Basic equations

Consider the fully developed Poiseuille flow in a circular pipe, where the velocity field is given by:

$$u(r) = 2u_m\left(1 - \frac{r^2}{R^2}\right) \quad (1)$$

and $u_m$ is the mean velocity on the pipe section. The time dependent energy equation can then be written:

$$\frac{\partial T}{\partial t} + 2u_m\left(1 - \frac{r^2}{R^2}\right)\frac{\partial T}{\partial x} = \alpha\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \quad (2)$$

with the usual assumption of neglecting the axial conduction in the fluid, which is justified for $Pe = PrRe \gg 1$.

Let split the temperature field into a steady (time averaged) and a fluctuating part and introduce the Fourier transform of the latter by: $T(x,r,t) = T_a(x,r) + \int_{-\infty}^{+\infty} S(x,r,\omega)e^{i\omega t}d\omega$. This also splits the energy equation into two equations:

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doi:10.1016/j.ijheatmasstransfer.2009.02.028

**Nomenclature**

| | |
|---|---|
| $a_n$ | complex variable: $a_n = \left\{\frac{1}{2} - \frac{(\lambda_n^2-\beta)}{4\lambda_n}\right\}$ |
| $c_n$ | non-dimensional phase velocity |
| $c, d$ | real coefficients |
| $D$ | pipe diameter |
| $f_n, g_n$ | coefficients |
| $Gz$ | Graetz number $Gz = \frac{x}{DRePr}$ |
| $h$ | convective coefficient |
| $k$ | thermal conductivity |
| $Pe$ | Peclet number: $Pe = RePr$ |
| $Pr$ | Prandtl number |
| $p_n$ | coefficients |
| $Q$ | heat flux Fourier transform |
| $q$ | heat flux |
| $R$ | pipe radius |
| $r$ | radial coordinate |
| $Re$ | Reynolds number: $Re = \frac{u_m D}{\nu}$ |
| $S$ | temperature Fourier transform |
| $T$ | temperature |
| $t$ | time |
| $u$ | axial flow velocity |
| $u_m$ | bulk velocity |
| $v_p$ | phase velocity |
| $W, Z$ | eigenfunctions |
| $x$ | axial coordinate |

*Greek symbols*

| | |
|---|---|
| $\alpha$ | thermal diffusivity |
| $\beta$ | imaginary non-dimensional frequency: $\beta = \frac{i\omega R^2}{\alpha}$ |
| $\gamma$ | complex number |
| $\delta$ | Dirac delta-function |
| $\eta, \zeta$ | non-dimensional coordinates |
| $\Theta$ | non-dimensional transformed temperature |
| $\lambda$ | square root of eigenvalue |
| $\nu$ | kinematic viscosity |
| $\Phi$ | Kummer function |
| $\varphi$ | phase delay |
| $\omega$ | pulsating frequency |

*Indexes*

| | |
|---|---|
| $a$ | time averaged |
| $fd$ | fully developed |
| $i$ | imaginary part of complex number |
| $n$ | order index |
| $r$ | real part of complex number |
| $w$ | wall surface |
| $0$ | at $x = 0$ |
| $*$ | complex conjugate |

*Other symbols*

| | |
|---|---|
| $\langle , \rangle$ | scalar product: $\langle f, g\rangle = \int_0^1 f(\eta)g(\eta)(1-\eta^2)\eta d\eta$ |
| $[ , ]$ | scalar product: $[f, g] = \int_0^1 f(\eta)g(\eta)\eta d\eta$ |

$$2u_m\left(1-\frac{r^2}{R^2}\right)\frac{\partial T_a}{\partial x} = \alpha\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_a}{\partial r}\right) \tag{3}$$

$$i\omega S + 2u_m\left(1-\frac{r^2}{R^2}\right)\frac{\partial S}{\partial x} = \alpha\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial S}{\partial r}\right)$$

the first one for the time averaged temperature field and the second one for the transformed temperature field. As the present study is devoted to the analysis of the fluctuating temperature field, only the second equation will be considered in the rest of the paper, while solutions of the first one are well known at least for the cases of imposed uniform wall temperature and imposed uniform wall heat flux [8]. Introducing the non-dimensional variables: $\eta = \frac{r}{R}; \zeta = \frac{x}{RRePr} = \frac{2}{Gz}; \beta = \frac{i\omega R^2}{\alpha} = i\beta_0$ the transformed equation becomes:

$$\beta S + (1-\eta^2)\frac{\partial S}{\partial \zeta} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial S}{\partial \eta}\right) \tag{4}$$

The symmetry condition at $\eta = 0$ imposes: $\left(\frac{\partial S}{\partial \eta}\right)_{\eta=0} = 0$, whereas two kinds of boundary conditions at $\eta = 1$ will be considered, namely uniform and constant temperature (i.e. nil temperature fluctuation amplitude at the wall: $S(\zeta, 1, \omega) = 0$) and uniform and constant wall heat flux (i.e. nil heat flux fluctuation amplitude at the wall: $\left(\frac{\partial S}{\partial \eta}\right)_{\eta=1} = 0$). Eq. (4) together with the boundary conditions previously defined set a Sturm–Liouville problem with complex eigenvalues. It should be pointed out that when the transformed field is known the temperature fluctuation field can be obtained by the inversion of the Fourier transform (numerical routines are also readily available), while the particular case of harmonic fluctuations can be analytically found by setting $S(\zeta, \eta, \omega) = \delta(\omega - \omega_0)S(\zeta, \eta)$.

## 3. The eigenvalue problem

Consider Eq. (4), a separation of variable approach allows to write the solution under the form: $S = e^{-\lambda^2\zeta}W(\lambda, \eta)$, where $W(\lambda, \eta)$ satisfies the ODE:

$$\frac{d}{d\eta}\left(\eta\frac{dW}{d\eta}\right) + [\lambda^2\eta(1-\eta^2) - \beta\eta]W = 0 \tag{5}$$

Making the following changes of variable: $z = \gamma\eta^2$ (with $\gamma^2 = \lambda^2$), Eq. (5) transforms to a particular form of the confluent hypergeometric equation, and the solution (finite at $\eta = 0$) is then:

$$W(\eta) = e^{-\frac{\gamma\eta^2}{2}}\Phi\left(\left\{\frac{1}{2} - \frac{(\lambda^2-\beta)}{4\gamma}\right\}, 1, \gamma\eta^2\right) \tag{6}$$

where $\Phi(a, b, z)$ is the Kummer function (see [9]). Setting $\lambda_0 = \sqrt{\lambda^2}, |\arg(\lambda_0)| < \frac{\pi}{2}$ (i.e. $Re\{\lambda_0\} > 0$) the two solutions of equation $\gamma^2 = \lambda^2$ are: $\gamma^+ = \lambda_0; \gamma^- = -\lambda_0$ yielding two possible solutions of Eq. (5), precisely:

$$W^+ = e^{-\frac{\lambda_0\eta^2}{2}}\Phi\left(\left\{\frac{1}{2} - \frac{(\lambda^2-\beta)}{4\lambda_0}\right\}, 1, \lambda_0\eta^2\right);$$

$$W^- = e^{\frac{\lambda_0\eta^2}{2}}\Phi\left(\left\{\frac{1}{2} + \frac{(\lambda^2-\beta)}{4\lambda_0}\right\}, 1, -\lambda_0\eta^2\right) \tag{7}$$

The Kummer identity (see [10]):

$$\Phi(a, b, z) = e^z\Phi(b-a, b, -z) \tag{8}$$

shows that the two solutions, corresponding to the two possible choices of $\gamma$, are actually identical.

The two B.C. on $\eta = 1$ now become:

$$W(\lambda, 1) = 0 \quad \text{or} \quad \left(\frac{\partial W}{\partial \eta}\right)_{\eta=1} = 0 \tag{9}$$

and the eigenvalues $\lambda_n^2$ are defined by the solutions of these equations. In the following it may be convenient to work with the square roots of the eigenvalues (namely $\lambda_n$) that satisfy two important conditions (see Appendix A):

$$Re\{\lambda_n\}^2 > Im\{\lambda_n\}^2; \quad sign[Re\{\lambda_n\}Im\{\lambda_n\}] = sign[\beta_0] \tag{10}$$

**Table 1**
Real and imaginary parts of the first eigenvalues $\lambda_n^2$ for isothermal and isoflux conditions and two values of non-dimensional frequency.

| n | Isothermal B.C. | | | | Isoflux B.C. | | | |
|---|---|---|---|---|---|---|---|---|
| | $\beta_0 = 1$ | | $\beta_0 = 10$ | | $\beta_0 = 1$ | | $\beta_0 = 10$ | |
| | $Re\{\lambda_n^2\}$ | $Im\{\lambda_n^2\}$ | $Re\{\lambda_n^2\}$ | $Im\{\lambda_n^2\}$ | $Re\{\lambda_n^2\}$ | $Im\{\lambda_n^2\}$ | $Re\{\lambda_n^2\}$ | $Im\{\lambda_n^2\}$ |
| 1 | 7.315769 | 1.251203 | 7.52976 | 12.48965 | 0.0416803 | 1.998956 | 4.260467 | 18.70500 |
| 2 | 44.610206 | 1.456185 | 44.68587 | 14.57230 | 25.66886 | 2.000838 | 24.554495 | 21.08419 |
| 3 | 113.92123 | 1.537144 | 113.9404 | 15.37660 | 83.85636 | 2.000128 | 83.293207 | 20.13379 |
| 4 | 215.24056 | 1.584583 | 215.2424 | 15.84837 | 174.1633 | 2.000038 | 173.82015 | 20.03922 |
| 5 | 348.56406 | 1.617082 | 348.5596 | 16.17220 | 296.5339 | 2.000016 | 296.29544 | 20.01607 |
| 6 | 513.88999 | 1.641315 | 513.8832 | 16.41397 | 450.9454 | 2.000007 | 450.767037 | 20.00792 |
| 7 | 711.21745 | 1.660378 | 711.2098 | 16.60430 | 637.3859 | 2.000004 | 637.245989 | 20.00440 |
| 8 | 940.54597 | 1.675938 | 940.5382 | 16.75972 | 855.8483 | 2.000002 | 855.734788 | 20.00266 |

As explained in the Appendix A, we can consider only those values of $\lambda_n$ such that $Re\{\lambda_n\} > 0$, which implies that: $sign[Im\{\lambda_n\}] = sign[\beta_0]$ and this simplifies the numerical search of the eigenvalues. To notice that $\lambda_n(-\beta_0) = \lambda_n^*(\beta_0)$ (see again the Appendix A) and this allows to search the eigenvalues only for $\beta_0 > 0$. The complex parameters $\lambda_n$ were calculated by means of purposely developed numerical routines (used to solve numerically Eq. (9) in the plane $\lambda_r > 0 \cap \lambda_i > 0$). In the present case this procedure was performed choosing an accuracy always better than $10^{-6}$ and Table 1 reports the first eight eigenvalues for different values of $\beta_0$ and for the two different boundary conditions. The three first eigenvalues can be compared to those reported by Unsal [7] finding a perfect agreement. It must be noticed that available symbolic computational tools can also be used to find the eigenvalues with high accuracy.

The most general solution can then be found under the form:

$$S(\zeta,\eta,\omega) = \sum_{n=1}^{\infty} S_n(\omega) e^{-\lambda_n^2\zeta} W_n(\eta) \tag{11}$$

where the values of $S_n$ are determined by the B.C. at $\zeta = 0$ ($S(0,\eta,\omega) = S_0(\eta,\omega)$) and due to the orthogonality of the functions $W_n$ we have:

$$S_n(\omega) = \frac{\int_0^1 S(0,\eta,\omega) W_n(\eta)\eta(1-\eta^2)d\eta}{\int_0^1 W_n(\eta)W_n(\eta)\eta(1-\eta^2)d\eta} = \frac{\langle S_0, W_n\rangle}{\langle W_n, W_n\rangle} \tag{12}$$

Let now consider separately the two problems set by the two B.C. at $\eta = 1$.

### 3.1. The uniform temperature case

In this case the first of (9) holds. As an example consider the special case of a uniformly distributed fluctuation of the temperature at $\zeta = 0$ (i.e. $S(0,\eta,\omega) = S_0(\omega)$). The coefficients $S_n$ are now $S_n(\omega) = S_0(\omega)f_n$ where:

$$f_n(\beta) = \frac{\langle 1, W_n\rangle}{\langle W_n, W_n\rangle} \tag{13}$$

and Table 2 shows that $|f_n|$ are all decreasing with $n$. The solution is then:

$$S(\zeta,\eta,\omega) = S_0(\omega)\sum_{n=1}^{\infty} f_n(\beta) e^{-\lambda_n^2\zeta} W_n(\eta) \tag{14}$$

Defining the average of $S$ over the section as:

$$S_m(\zeta,\omega) = \frac{\int_0^1 S(\zeta,\eta,\omega)u_r(\eta)\eta d\eta}{\int_0^1 u_r(\eta)\eta d\eta} = 4\langle S, 1\rangle \tag{15}$$

we get

$$S_m(\zeta,\omega) = 4S_0(\omega)\sum_{n=1}^{\infty} e^{-\lambda_n^2\zeta} f_n^2 \langle W_n W_n\rangle \tag{16}$$

To notice that the obvious condition: $S_m(0,\eta,\omega) = S_0(\omega)$ imposes $4\sum_{n=1}^{\infty} f_n^2\langle W_n W_n\rangle = 1$ which is a useful check of accuracy for the series truncation (for $\beta_0 < 50$ the partial sum of the first 8 terms gave an error lower than $10^{-5}$ for $\zeta > 0.02$). The ratio:

$$G(\zeta,\omega) = \frac{S_m(\zeta,\omega)}{S_m(0,\omega)} = \frac{S_m(\zeta,\omega)}{S_0(\omega)} = 4\sum_{n=1}^{\infty} e^{-\lambda_n^2\zeta} f_n^2\langle W_n W_n\rangle \tag{17}$$

gives the response of the average fluid temperature at location $\zeta$, it is then a transfer function for the mean section temperature fluctuation. In Fig. 1 the absolute value of $G$ is plotted versus the non-dimensional frequency $\beta_0$ for different downstream locations $\zeta$.

In analogy with the steady state solution [8], let define the non-dimensional temperature profile transform as:

$$\Theta = \frac{S(\zeta,\eta,\omega) - S(\zeta,1,\omega)}{S_m(\zeta,\omega) - S(\zeta,1,\omega)} = \frac{S(\zeta,\eta,\omega)}{S_m(\zeta,\omega)} \tag{18}$$

and substituting Eqs. (14) and (16) yields:

$$\Theta = \frac{\sum_{n=1}^{\infty} f_n(\beta) e^{-\lambda_n^2\zeta} W_n(\eta)}{4\sum_{n=1}^{\infty} e^{-\lambda_n^2\zeta} f_n^2\langle W_n W_n\rangle} \tag{19}$$

**Table 2**
Real and imaginary parts of the coefficients $f_n$ (for isothermal wall conditions) and $g_n$ (for isoflux conditions) and two values of non-dimensional frequency.

| n | Isothermal B.C. | | | | Isoflux B.C. | | | |
|---|---|---|---|---|---|---|---|---|
| | $\beta_0 = 1$ | | $\beta_0 = 10$ | | $\beta_0 = 1$ | | $\beta_0 = 10$ | |
| | $Re\{f_n\}$ | $Im\{f_n\}$ | $Re\{f_n\}$ | $Im\{f_n\}$ | $Re\{g_n\}$ | $Im\{g_n\}$ | $Re\{g_n\}$ | $Im\{g_n\}$ |
| 1 | 1.47658 | 0.00940 | 1.490744 | 0.092624 | 1.0016 | 0.062609 | 1.2956 | 0.72366 |
| 2 | −0.8063 | −0.01418 | −0.82217 | −0.14056 | −0.00149 | −0.08008 | −0.28772 | −0.9095 |
| 3 | 0.588775 | 0.007575 | 0.590636 | 0.075951 | −0.00016 | 0.025417 | −0.01385 | 0.26699 |
| 4 | −0.47588 | −0.00461 | 0.47596 | −0.04623 | 8.33E−05 | −0.01245 | 0.009094 | −0.12656 |
| 5 | 0.405005 | 0.003096 | 0.404827 | 0.030996 | −8.65E−05 | 0.00739 | −0.00499 | 0.074399 |
| 6 | −0.3558 | −0.00222 | −0.35561 | −0.02222 | 1.66E−05 | −0.00489 | 0.002858 | −0.0491 |
| 7 | 0.319143 | 0.001671 | 0.318989 | 0.016712 | −7.02E−05 | 0.00348 | −0.00183 | 0.034859 |
| 8 | −0.29079 | −0.00130 | −0.29067 | −0.01303 | −7.17E−06 | −0.002609 | 0.001153 | −0.02604 |

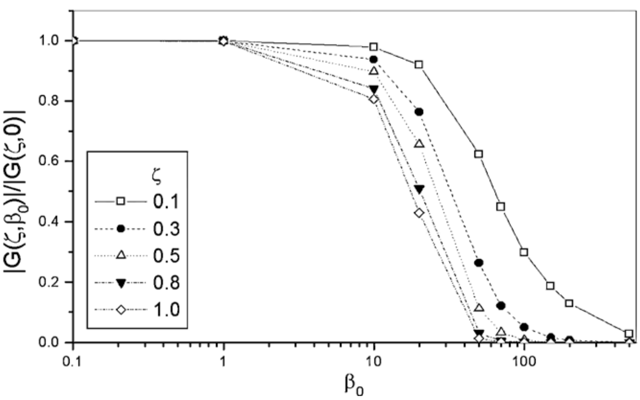


**Fig. 1.** Absolute value of the transfer function $G$ normalised by the limiting value at $\beta_0 = 0$.

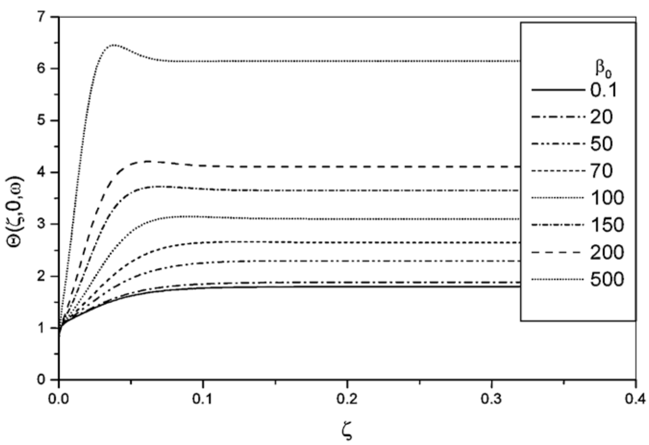


**Fig. 2.** Centreline value of the non-dimensional temperature transform ($\Theta(\zeta,0,\omega)$) for different values of the non-dimensional frequency $\beta_0$.

and Fig. 2 reports the centreline values ($\Theta(\zeta,0,\omega)$) of this function for different values of the parameter $\beta_0$, calculated using the first 8 terms of the series. Due to the presence of the exponential terms $e^{-\lambda_n^2\zeta}$ and the fact that $Re\{\lambda_n^2\} > 0$ both $S(\zeta,\eta,\omega)$ and $S_m(\zeta,\eta,\omega)$ tend to zero when $\zeta \to \infty$, but observing that (see Table 1) $Re\{\lambda_{n+1}^2\} > Re\{\lambda_n^2\}$, for $\zeta$ sufficiently large only the first term survives in both series and the limiting case is then:

$$\Theta_{fd}(\eta) = \frac{W_1(\eta)}{4f_1\langle W_1 W_1\rangle} \tag{20}$$

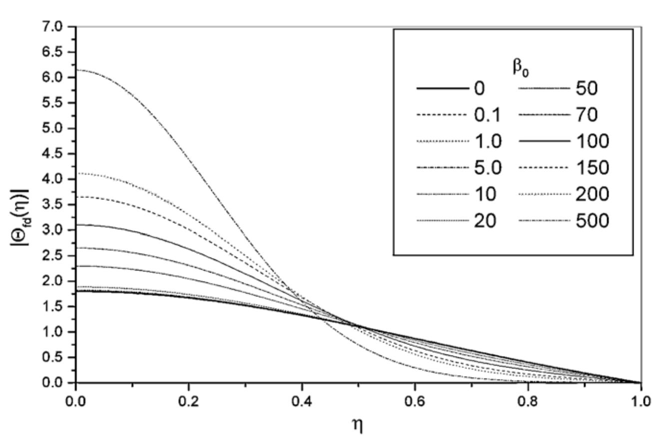


**Fig. 3.** Fully developed solution of the non-dimensional temperature transform profile for different values of the non-dimensional frequency $\beta_0$.

that gives the "fully developed" solution for the transformed field. Fig. 3 shows that an increase of the fluctuation frequency have the effect of confining the fluid temperature fluctuation towards the pipe axis.

Another important parameter is the ratio between the transformed wall heat flux $Q$ (defined by: $q_w(\zeta,t) = q_{wa}(\zeta) + \int_{-\infty}^{+\infty} Q(\zeta,\omega)e^{i\omega t}d\omega$) and the mean section temperature transform:

$$h(\zeta,\omega) = \frac{Q(\zeta,1,\omega)}{S_m(\zeta,\omega) - S(\zeta,1,\omega)} = \frac{Q(\zeta,1,\omega)}{S_m(\zeta,\omega)} \tag{21}$$

From the general solution and Fourier law:

$$Q(\zeta,1,\omega) = -\frac{k}{R}S_0(\omega)\sum_{n=1}^{\infty} f_n(\beta)e^{-\lambda_n^2\zeta}\left(\frac{dW_n(\eta)}{d\eta}\right)_{\eta=1} \tag{22}$$

where [9]

$$\left(\frac{dW_n(\eta)}{d\eta}\right)_{\eta=1} = -\lambda_n e^{-\frac{\lambda_n\eta^2}{2}}[\Phi(a_n,1,\lambda_n) - 2a_n\Phi(a_n+1,2,\lambda_n)] \tag{23}$$

with $a_n = \left\{\frac{1}{2} - \frac{(\lambda_n^2-\beta)}{4\lambda_n}\right\}$. The non-dimensional form of this parameter brings an interesting relation with the Nusselt number as it will be seen below. Define:

$$Nu(\zeta,\omega) = \frac{h(\zeta,\omega)D}{k} = -\frac{1}{2}\frac{\sum_{n=1}^{\infty} f_n(\beta)e^{-\lambda_n^2\zeta}W_n'(1)}{\sum_{n=1}^{\infty} e^{-\lambda_n^2\zeta}f_n^2\langle W_n W_n\rangle} \tag{24}$$

to obtain a meaningful result, let integrate Eq. (5) between $\eta = 0$ and $\eta = 1$, then:

$$W_n'(1) = \left(\frac{dW_n(\eta)}{d\eta}\right)_{\eta=1} = (\beta p_n - \lambda_n^2 f_n)\langle W_n, W_n\rangle \tag{25}$$

with:

$$p_n = \frac{\int_0^1 W_n(\eta)\eta d\eta}{\langle W_n, W_n\rangle} = \frac{[W_n, 1]}{\langle W_n, W_n\rangle} \tag{26}$$

and substituting into Eq. (24) one obtains:

$$Nu(\zeta,\omega) = -\frac{1}{2}\frac{\sum_{n=1}^{\infty} f_n(\beta)e^{-\lambda_n^2\zeta}(\beta p_n - \lambda_n^2 f_n)\langle W_n, W_n\rangle}{\sum_{n=1}^{\infty} e^{-\lambda_n^2\zeta}f_n^2\langle W_n W_n\rangle} \tag{27}$$

and Fig. 4 reports the numerical evaluation using the first 8 terms of the series. Again for sufficiently large values of $\zeta$ only the first term of the series survives and the "fully developed" value of $Nu(\omega)$ is then:

$$Nu_{fd} = \left[\frac{(\lambda_{1,r})^2 - (\lambda_{1,i})^2}{2} + \beta_0\frac{(p_{1,i}f_{1,r} - p_{1,r}f_{1,i})}{2|f_1|^2}\right] + i\left[\lambda_{1,r}\lambda_{1,i} - \beta_0\frac{(p_{1,r}f_{1,r} + p_{1,i}f_{1,i})}{2|f_1|^2}\right] \tag{28}$$

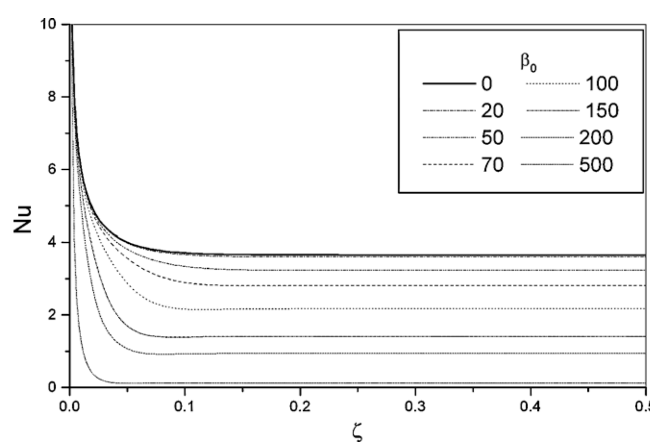


**Fig. 4.** Values of $Nu(\zeta,\omega)$ for different values of the non-dimensional frequency $\beta_0$.

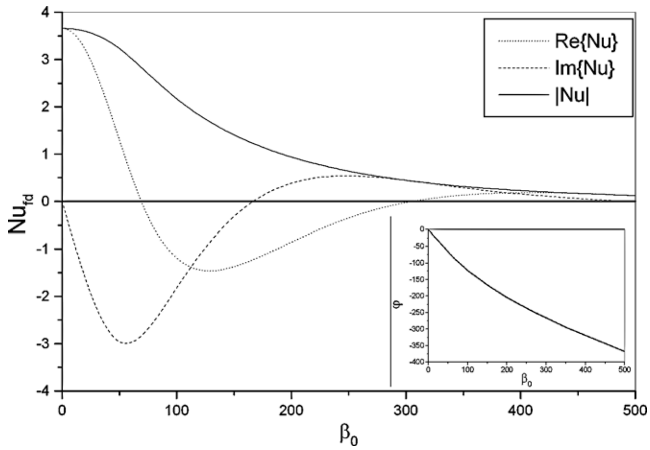


**Fig. 5.** Fully developed values of $Nu$ versus non-dimensional frequency $\beta_0$.

It is worth to notice that the limiting value $\beta = 0$ gives the usual value of Nusselt number found for the steady problem (with uniform wall temperature) $Nu_{fd} = 3.656$, while Fig. 5 shows the values of $Nu_{fd}$ for different values of $\beta_0$. To notice finally that the phase difference between the harmonic fluctuation of the wall heat flux and the mean section temperature can be written as:

$$\varphi = \arctan\left(\frac{Im\{Nu\}}{Re\{Nu\}}\right) \tag{29}$$

and the results are also reported in Fig. 5. From its definition, $Nu(\zeta, \omega)$ gives the response in term of wall heat flux to the fluctuation of the mean temperature at the given location, it is then consistent that the increase of the non-dimensional frequency $\beta_0$ decreases the response amplitude and increases the phase lag.

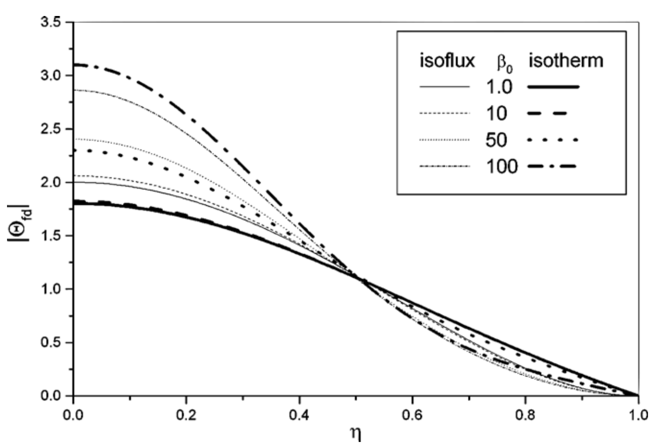


**Fig. 6.** Fully developed radial distribution of the non-dimensional temperature transform profile for different values of the non-dimensional frequency $\beta_0$ for the case of constant and uniform wall heat flux and wall temperature.

*3.2. The uniform wall heat flux*

Consider now the case of imposed constant and uniform heat flux on the wall (which for $q_r = 0$ also represents the case of adiabatic wall). The B.C. for the transformed field is the second of (9) that can be written explicitly [9]:

$$-\Phi(a_n, 1, \lambda) + 2a_n\Phi(a_n + 1, 2, \lambda) = 0 \tag{30}$$

The eigenvalues $\lambda_n'^2$ yielded by this equation are reported in Table 1 for two values of $\beta_0$. The eigenfunctions: $Z_n(\eta) = e^{-\frac{\lambda_n'\eta^2}{2}}\Phi[a_n, 1, \lambda_n'\eta^2]$ are different from the $W_n$ (as the eigenvalues are different) but the solution of Eq. (4) has still the general form:

$$S(\zeta, \eta, \omega) = \sum_{n=1}^{\infty} S_n'(\omega)e^{-\lambda_n'^2\zeta}Z_n(\eta) \tag{31}$$

where the coefficients $S_n'$ are defined by the inlet condition. Considering again the case of uniform temperature distribution at the inlet section (i.e. $S(0, \eta, \omega) = S_0(\omega)$) and defining: $g_n = \frac{\langle 1, Z_n\rangle}{\langle Z_n, Z_n\rangle}$ (Table 2 reports some values) the non-dimensional transformed temperature profile defined by Eq. (18) and the "fully developed" solution are now:

$$\Theta = \frac{\sum_{n=1}^{\infty} g_n(\beta)e^{-\lambda_n'^2\zeta}[Z_n(\eta) - Z_n(1)]}{\sum_{n=1}^{\infty} g_n(\beta)e^{-\lambda_n'^2\zeta}[4g_n\langle Z_n Z_n\rangle - Z_n(1)]}; \quad \Theta_{fd} = \frac{Z_1(\eta) - Z_1(1)}{4g_1\langle Z_1 Z_1\rangle - Z_1(1)} \tag{32}$$

and the last one is reported in Fig. 6 together with the uniform wall temperature case for comparison, showing a variation with $\beta$ qualitatively similar but with consistent quantitative differences.

As a general statement one can say that from the above reported results, the fully developed region for the transformed field is reached for $Gz < 20$ (as for the time independent case), at least for all the values of $\beta_0$ lower that 500.

## 4. The harmonic case

It is of a certain interest to analyse the particular case of harmonic inlet temperature variation. In this case the inlet conditions become: $S_0(\omega') = S_0\delta(\omega' - \omega)$ and the temperature fluctuation field is:

$$T'(\zeta, \eta, t) = Re\{S(\zeta, \eta, \omega)e^{i\omega t}\}$$
$$= Re\left\{\sum_{n=1}^{\infty} S_n(\omega)P_n(\eta)e^{-(\lambda_{n,r}^2 - \lambda_{n,i}^2)\zeta}e^{i(\omega t - 2\lambda_{n,r}\lambda_{n,i}\zeta)}\right\} \tag{33}$$

where $P_n(\eta)$ are the eigenfunctions. Eq. (33) shows that the fluctuating temperature field is an overlapping of thermal waves with amplitude decreasing with $\zeta$ and with phase velocity equal to

$$v_{p,n} = \frac{\beta_0}{\lambda_{n,r}\lambda_{n,i}}u_m = c_n u_m \tag{34}$$

**Table 3**
Values of the coefficients $c_n$ for the isothermal and isoflux case and some values of the non-dimensional frequency.

| $n$ | Isothermal B.C. | | | | Isoflux B.C. | | | |
|---|---|---|---|---|---|---|---|---|
| | $\beta_0 = 1$ | $\beta_0 = 10$ | $\beta_0 = 50$ | $\beta_0 = 100$ | $\beta_0 = 1$ | $\beta_0 = 10$ | $\beta_0 = 50$ | $\beta_0 = 100$ |
| 1 | 1.59846 | 1.60133 | 1.66389 | 1.75192 | 1.00052 | 1.06923 | 1.68222 | 1.75451 |
| 2 | 1.37345 | 1.37247 | 1.35679 | 1.39903 | 0.99958 | 0.94858 | 1.28018 | 1.44931 |
| 3 | 1.30111 | 1.30068 | 1.28947 | 1.25211 | 0.99994 | 0.99336 | 0.64478 | 1.28076 |
| 4 | 1.26216 | 1.26196 | 1.25686 | 1.23780 | 0.99998 | 0.99804 | 0.97160 | 1.13522 |
| 5 | 1.23680 | 1.23669 | 1.23408 | 1.22505 | 0.99999 | 0.99920 | 0.98184 | 1.02083 |
| 6 | 1.21854 | 1.21847 | 1.21698 | 1.21205 | 1.00000 | 0.99960 | 0.99083 | 0.992590 |
| 7 | 1.20455 | 1.20451 | 1.20359 | 1.20061 | 1.00000 | 0.99978 | 0.99484 | 0.991832 |
| 8 | 1.19336 | 1.19334 | 1.19273 | 1.19081 | 1.00000 | 0.99987 | 0.99684 | 0.993735 |

the evaluation of $c_n$ shown in Table 3, shows that its value is always lower than 2 (as expected since the maximum fluid velocity is $2u_m$) and the dependence of $c_n$ on the frequency shows the dispersive character of such waves.

## 5. Conclusions

An analytical solution of the periodic temperature field in a fully developed pipe flow induced by periodic (of any shape and not necessarily harmonic) fluctuation of the inlet temperature was given in closed form in terms of a series of mutually orthogonal Kummer functions. The complex eigenvalue problems set by the two kind of B.C. (namely, constant and uniform wall temperature and wall heat flux) were analysed and the dependence of the eigenvalues on the non-dimensional frequency $\beta_0 = \frac{\omega R^2}{\alpha}$ was reported. The conditions for a fully developed temperature fluctuation field were found and the transformed temperature field was given in analytical form. The fluctuating temperature field was also found to be represented by the overlapping of dispersive "thermal waves" travelling downstream with phase velocity depending on the frequency and on the order.

## Appendix A

Consider the eigensolutions $P_n(\eta) = e^{-\frac{\lambda_n \eta^2}{2}} \Phi(a(\lambda_n), 1, \lambda_n \eta^2)$ satisfying the ODE:

$$\frac{d}{d\eta}\left(\eta \frac{dP_n}{d\eta}\right) + [\lambda_n^2 \eta (1-\eta^2) - \beta\eta] P_n = 0 \tag{35}$$

with B.C. $P_n(1) = 0$ or $P'_n(1)$, then multiplying the ODE by $P_n^*(\eta)$ and integrating between 0 and 1

$$\lambda_n^2 = \frac{\left[\frac{dP_n}{d\eta}, \frac{dP_n^*}{d\eta}\right]}{\langle P_n P_n^* \rangle} + \beta \frac{[P_n, P_n^*]}{\langle P_n P_n^* \rangle} - \frac{P_n^*(1)\left(\frac{dP_n}{d\eta}\right)_{\eta=1}}{\langle P_n P_n^* \rangle} \tag{36}$$

The last term is always nil when one of the B.C. conditions hold, thus showing that $Re\{\lambda_n^2\} > 0$ and that $sign[Im\{\lambda_n^2\}] = sign[\beta_0]$, and incidentally, for $\beta_0 = 0$ all the eigenvalues are real and positive. These results imply also the following conditions on the real and imaginary parts of $\lambda_n$ : $Re\{\lambda_n\}^2 > Im\{\lambda_n\}^2$; and $[Re\{\lambda_n\} Im\{\lambda_n\}] = sign[\beta_0]$. Moreover, the eigenvalues $\lambda_n^2$ depend on the value of $\beta_0$ and the following relation holds:

$$\lambda_n(-\beta) = \lambda_n^*(\beta) \tag{37}$$

In fact, consider again Eq. (35), if $P = P(\beta_0, \lambda, \eta)$ is a solution, then taking the complex conjugate of Eq. (35) the following relation holds:

$$P(-\beta_0, \lambda^*, \eta) = P^*(\beta_0, \lambda, \eta) \tag{38}$$

Consider now the general form of linear B.C.

$$cP(\beta_0, \lambda, 1) + d\frac{\partial P(\beta_0, \lambda, 1)}{\partial \eta} = 0 \tag{39}$$

with $c, d$ real, (again the B.C. given above are obtained setting alternatively $d = 0$ or $c = 0$) that yields the parameters $\lambda_n = \lambda_n(\beta_0)$. Equation:

$$cP(-\beta_0, \lambda^*, 1) + d\frac{\partial P(-\beta_0, \lambda^*, 1)}{\partial \eta} = 0 \tag{40}$$

yields the parameters: $\lambda_n^* = \lambda_n^*(-\beta_0)$, but due to (38), Eq. (40) is identical to:

$$cP^*(\beta_0, \lambda, 1) + d\frac{\partial P^*(\beta_0, \lambda, 1)}{\partial \eta} = 0 \tag{41}$$

that in turn, due to the fact that $c$ and $d$ are real, has the same solutions of Eq. (39), then: $\lambda^*(-\beta_0) = \lambda(\beta_0)$ and (37) is proven.

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